Computing Classic Closeness Centrality, at Scale

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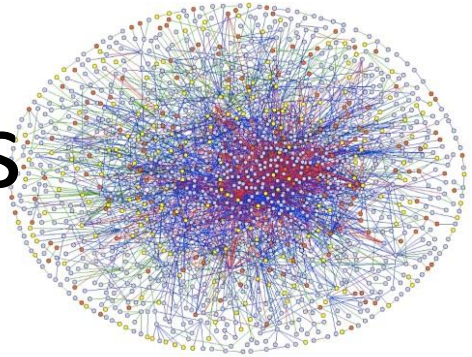
Very Large Graphs



- Model relations and interactions (edges) between entities (nodes)
  - Call detail, email exchanges,
  - Hyperlinks
  - Social Networks (friend, follow, like),
  - Commercial transactions,…
- Need for scalable analytics

Centrality

- Centrality of a node measures its importance. Applications: ranking, scoring, characterize network properties.
- Several structural centrality definitions:
  - **Betweeness:** effectiveness in connecting pairs of nodes
  - **Degree:** Activity level
  - **Eigenvalue:** Reputation
  - **Closeness:** Ability to reach/influence others.

Closeness Centrality

Importance measure of a node that is a function of the distances from a node to all other nodes.

Classic Closeness Centrality [(Bavelas 1950, Beaucahmp 1965, Sabidussi 1966)]

(Inverse of) the average distance to all other nodes

$$B^{-1}(v) = \frac{n-1}{\sum_{u \in V} d_{uv}}$$

Maximum centrality node is the *1-median*

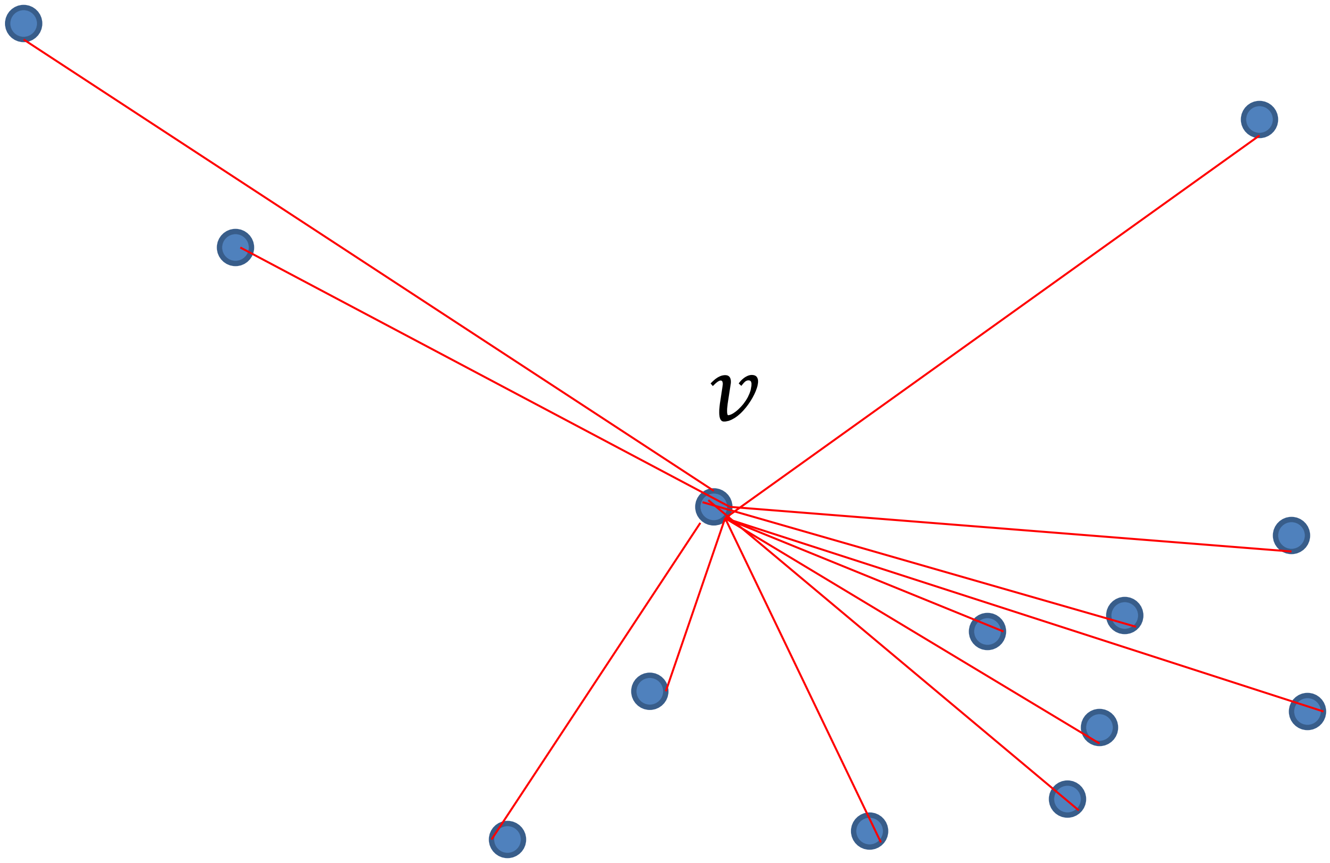
Computing Closeness Centrality

- ❑ Run Dijkstra's algorithm from source v.
- ❑ **Compute sum of distances $\sum_{u \in V} d_{uv}$ from v to all other nodes**

$$B(v) = \frac{\sum_{u \in V} d_{uv}}{n - 1}$$

!! Does not scale when we want $B(v)$ for many or all nodes in a large graph

Centrality of v using Dijkstra



Exact, but does not scale for many nodes on large graphs.

Goals

- ❑ Scalable algorithm to compute/estimate centrality scores of **all** nodes
- ❑ Accurate: Small relative error: within $(1+\epsilon)$ with high probability
- ❑ Scalable:
  - ❑ Processing cost $O(|G|)$ (can depend on ϵ^{-1})
  - ❑ Constant memory per node, independent of ϵ

Have to settle for approximation: Exact computation, even of the maximum centrality node (1-median) seems as hard as APSP **[Abboud Vassilevska-Williams 2015]**

Algorithmic Overview

- Approach I: ***Sampling***
  - Properties: good for “close” distances
- Approach II: ***Pivoting***
  - Properties: good for “far” distances
- ***Hybrid***: Best of all worlds

Approach I: Sampling

[EW 2001, OCL2008,Indyk1999,Thorup2001]

Computation

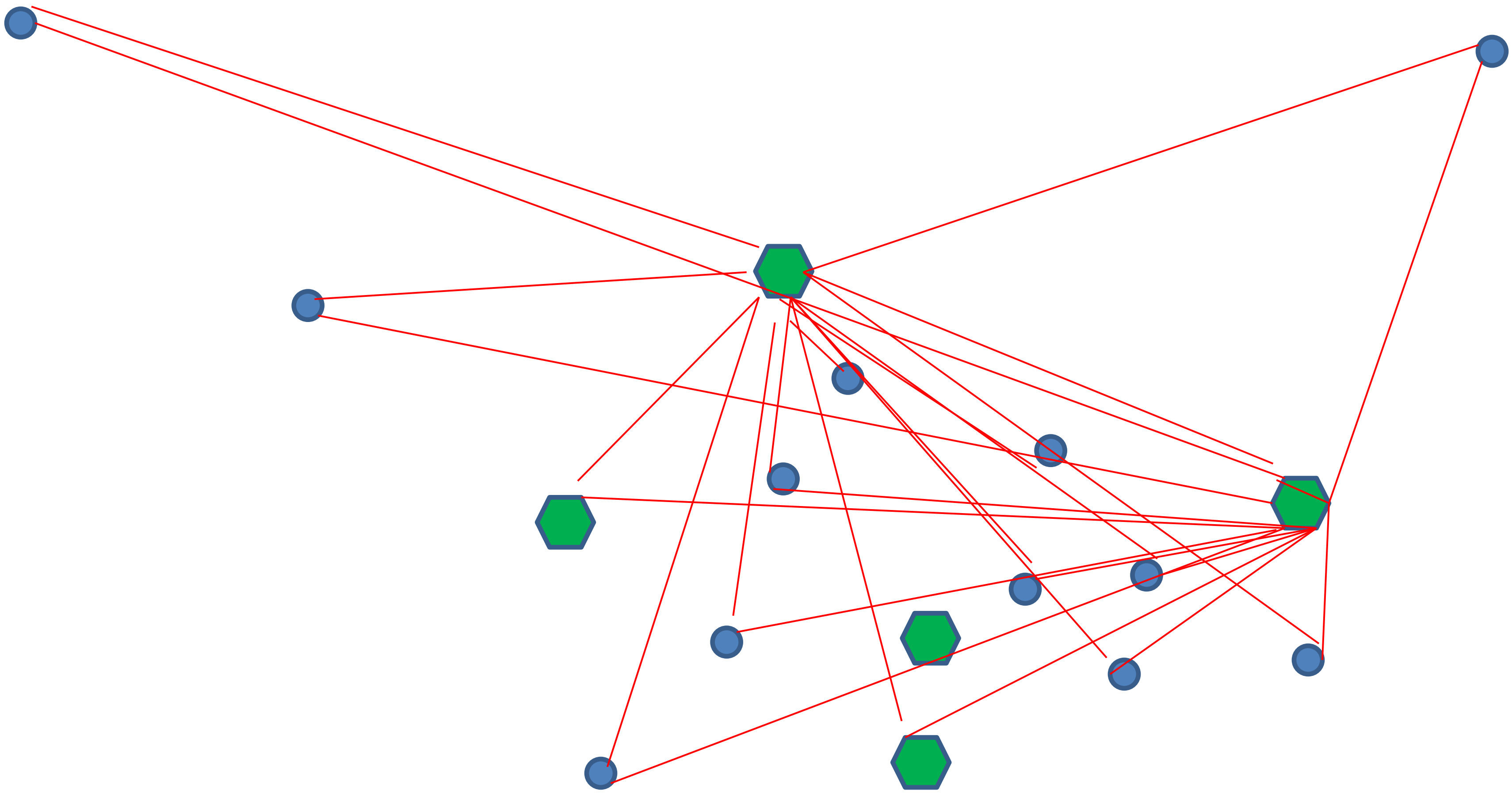
- uniform sample C of k nodes
- Ran Dijkstra from each $u \in C$ (Gives us exact $B(u)$ for $u \in C$)

Estimation

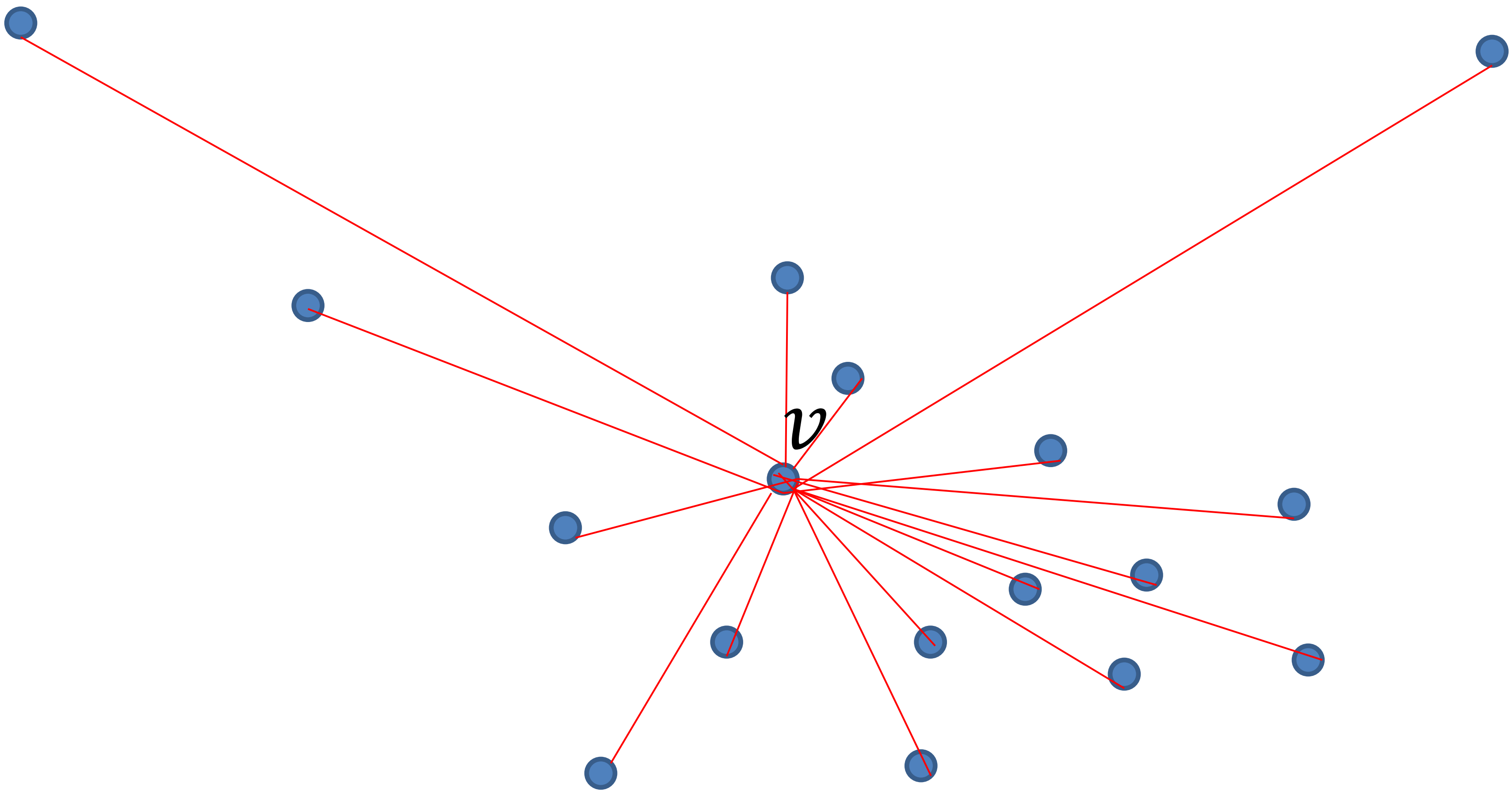
- For $v \in V \setminus C$ estimate $B(v)$ by the average distance to *sampled* nodes

$$\hat{B}(v) = \frac{\sum_{u \in C} d_{uv}}{k}$$

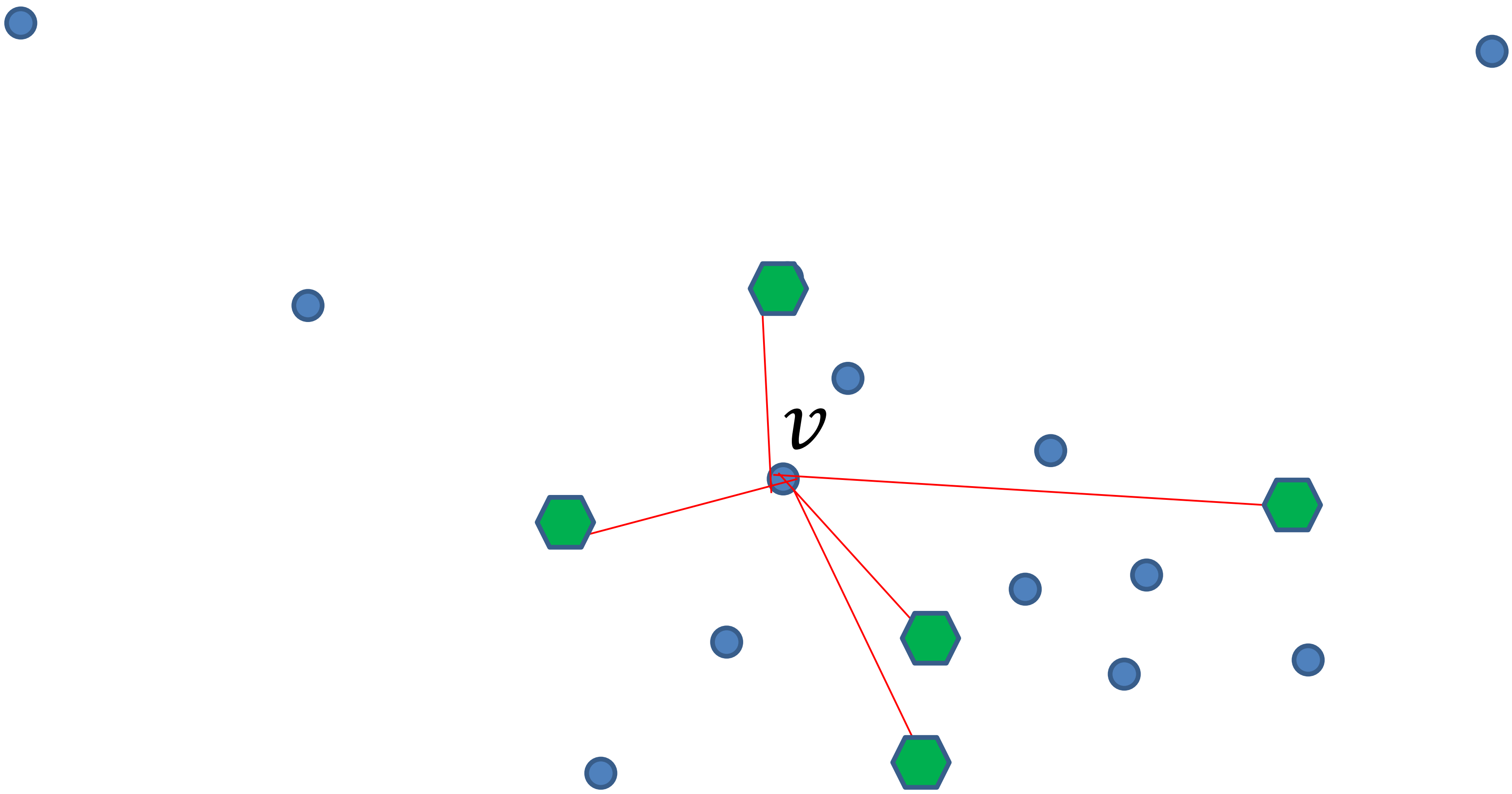
Sampling



$$B(v)\ ?$$

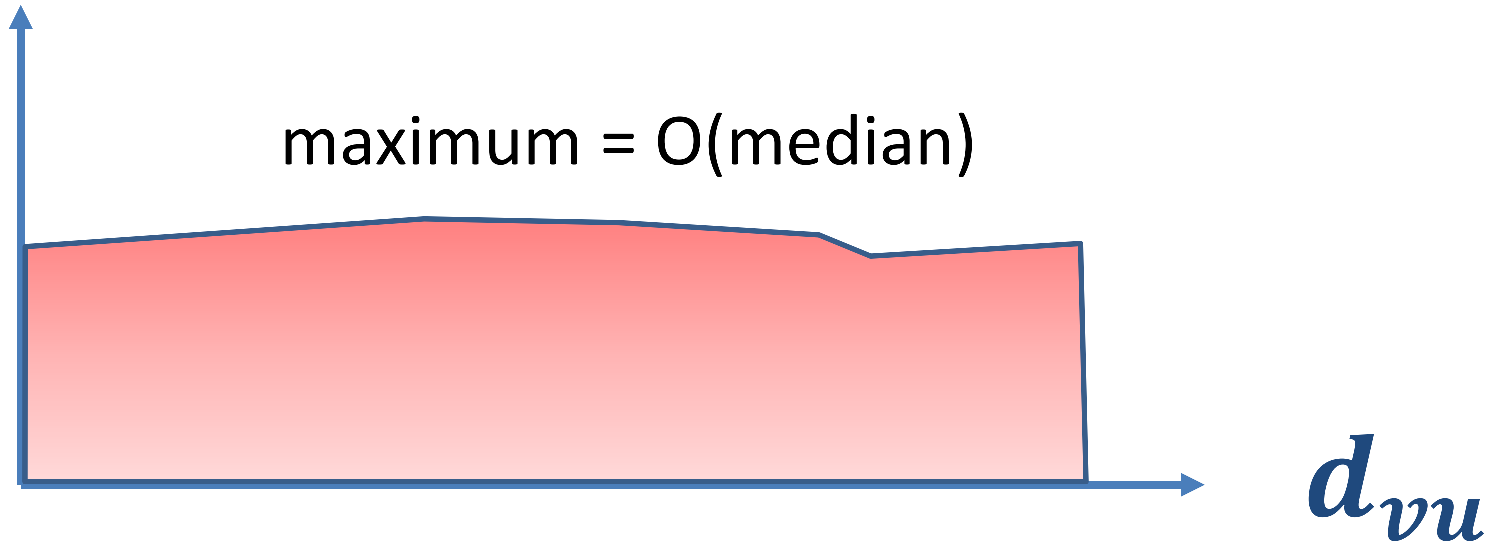


Sampling Estimator $\hat{B}(v)$



Sampling: Properties

- ❑ Unbiased
- ❑ Can have large variance -- uniform sample can miss heavy (far) items. Estimate quality depends on **distribution of distances from v**



Works! sample average concentrated population average

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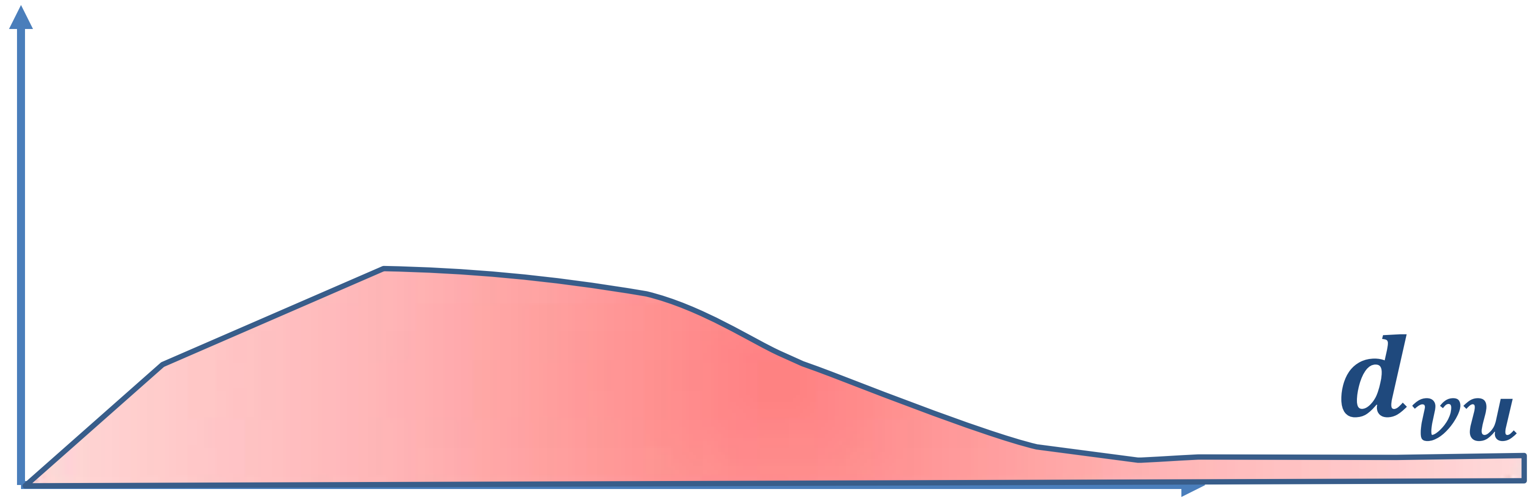


Heavy tail -- sample average has high variance – relative error

Approach II: Pivoting

Computation

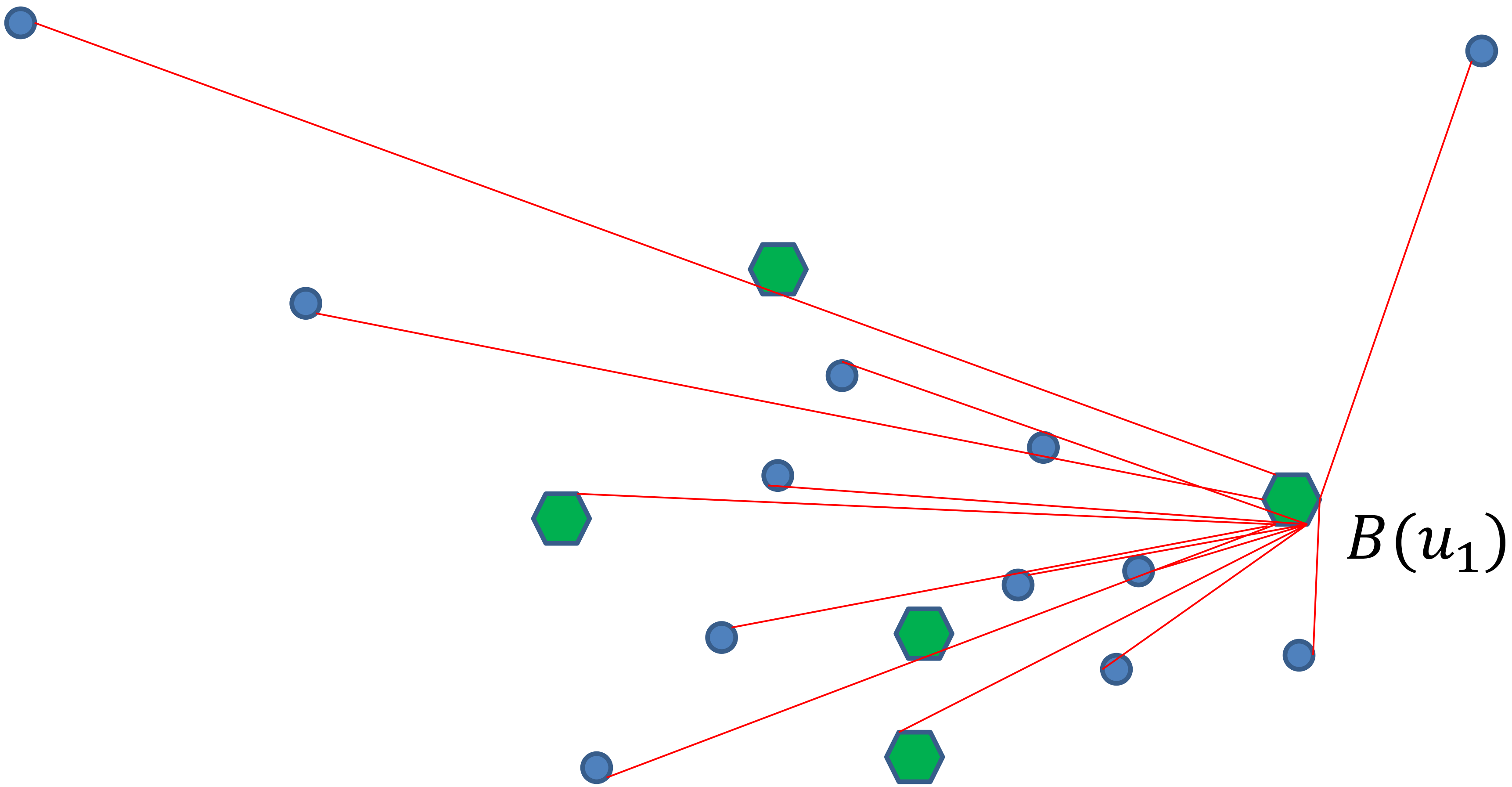
- uniform sample C of k nodes
- Ran Dijkstra from each $u \in C$ (Gives us exact $B(u)$ for $u \in C$)

Estimation

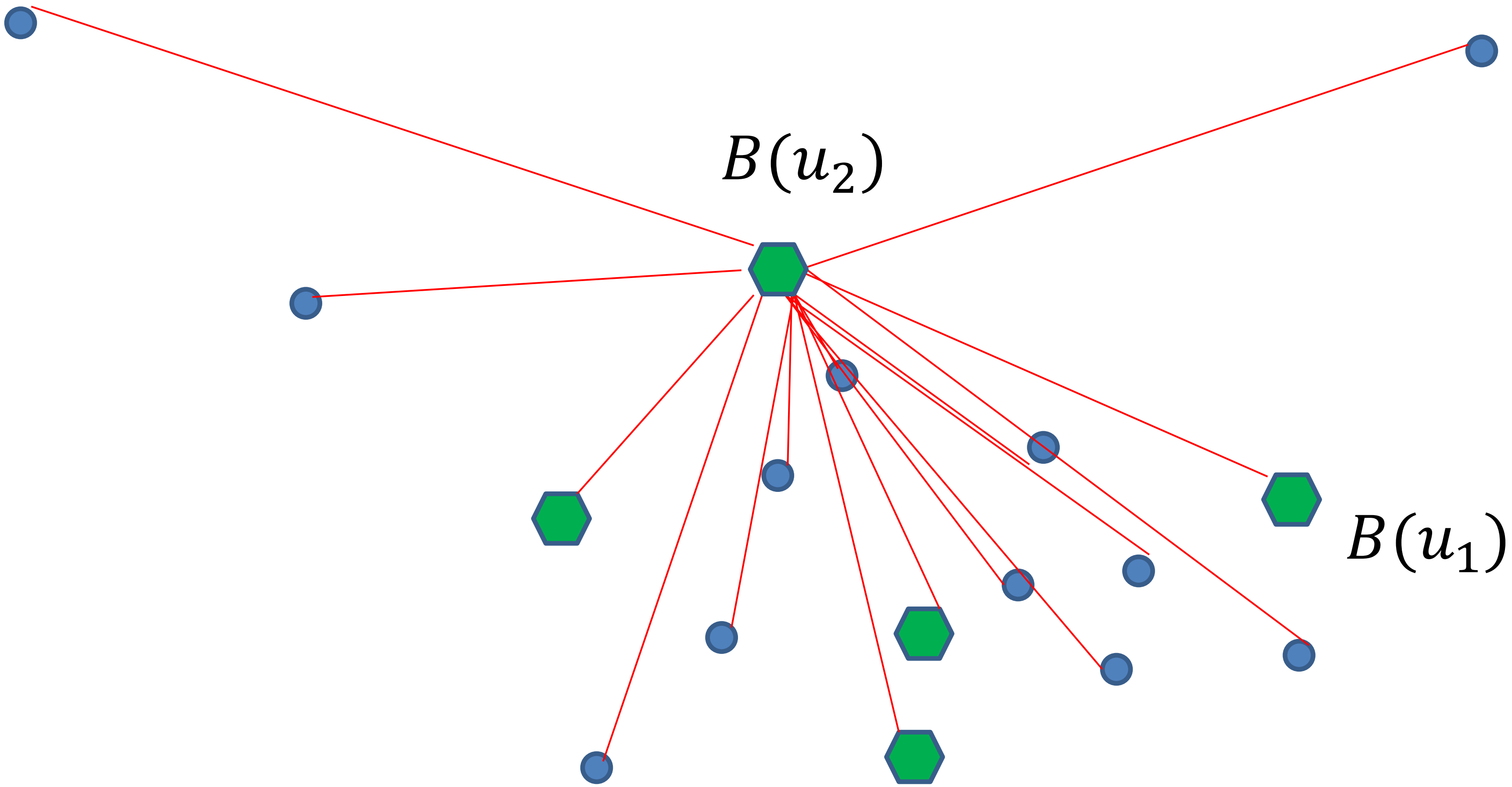
- For $v \in V \setminus C$, find closest sample node "pivot" $c(v) \in C$.
- estimate using pivot average distance

$$\hat{B}(v) = B(c(v))$$

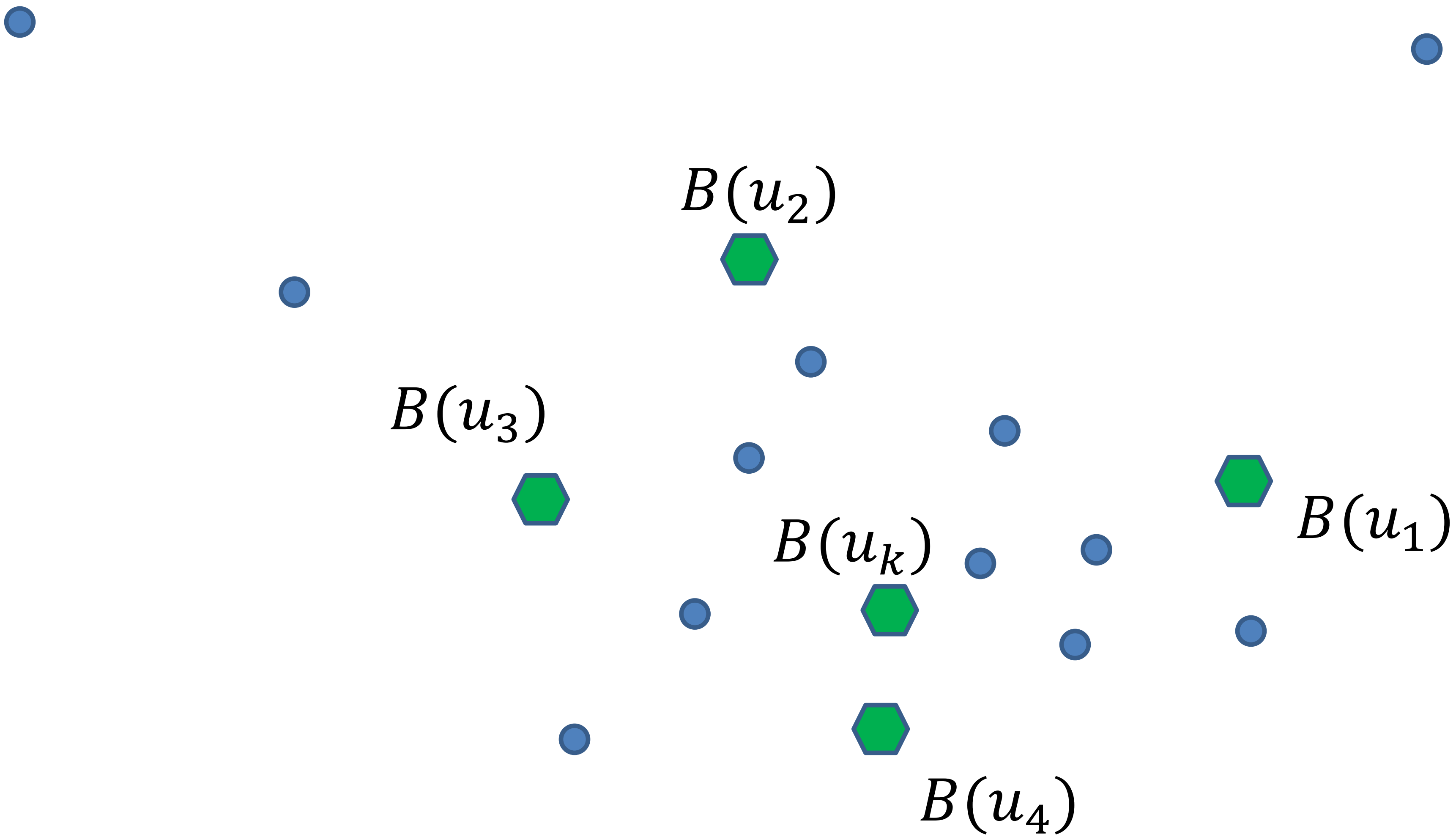
Pivoting



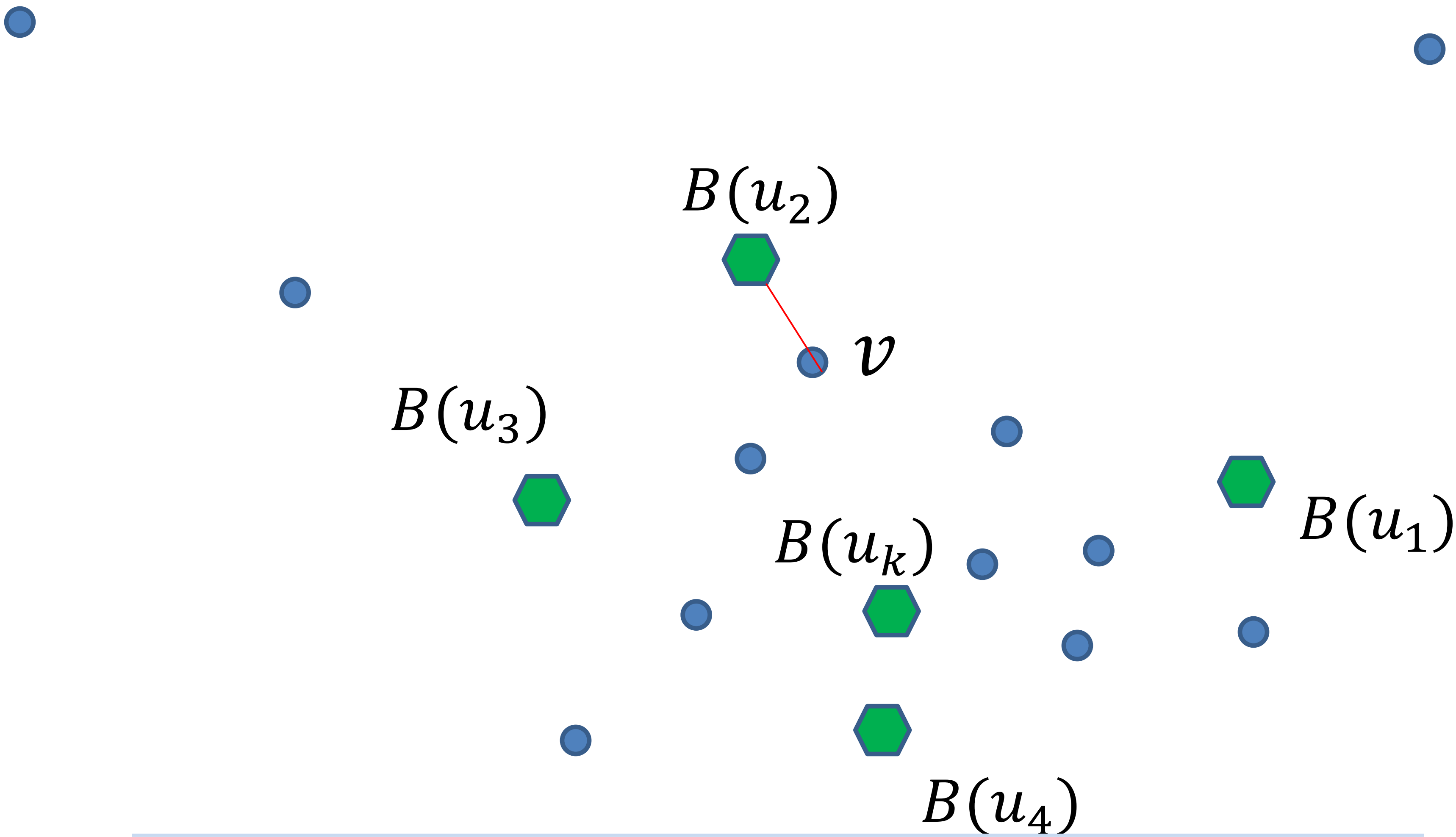
Pivoting



Pivoting



Pivoting $\hat{B}(v)$



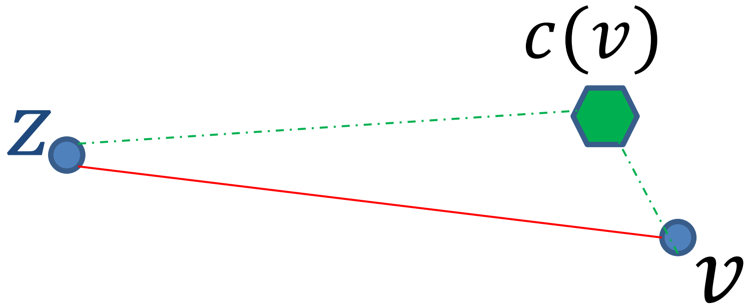
Inherit centrality of pivot (closest sampled node)

Pivoting: properties

❑Estimate is within $\pm d_{vc(v)}$ of true $B(v)$

<u>Proof:</u>

- triangle inequality: for all z,
  $$d_{c(v)z} - d_{vc(v)} \le d_{vz} \le d_{c(v)z} + d_{vc(v)}$$
- Therefore $\left|B(v) - B\big(c(v)\big)\right| \le d_{vc(v)}$



Pivoting: properties

- Estimate is within $\pm d_{vc(v)}$ of true $B(v)$
- *WHP upper bound* $\hat{B}(v) \equiv d_{vc(v)} + B\big(c(v)\big)$ satisfies

$$B(v) \leq \hat{B}(v) \leq 4B(v)$$

Proof: WHP pivot is one of the $\frac{n}{k}\log n$ closest nodes

$$\Rightarrow \quad B(v) \geq \left(1 - \frac{\log n}{k}\right) d_{vc(v)}$$

$$\hat{B}(v) = d_{vc(v)} + B(c(v)) \leq 2d_{vc(v)} + B(v)$$

Triangle inequality

WHP

$$\leq B(v) \cdot \left(1 + \frac{2}{\left(1 - \frac{\log n}{k}\right)}\right)$$

Pivoting: properties

- Estimate is within $\pm d_{vc(v)}$ of true $B(v)$
- *WHP upper bound* $\hat{B}(v) = d_{vc(v)} + \hat{B}(c(v))$ satisfies

$$B(v) \leq \hat{B}(v) \leq 4B(v)$$

Bounded relative error for any instance !

A property we could not obtain with sampling

Pivoting vs. Sampling

- Same computation/information:
  - k Dijkstras from a uniform sample
- Different properties on estimate quality
  - Sampling accurate when distance distribution is concentrated.
  - Pivoting accurate with heavier tail.

But neither gives us a small relative error !

$$\hat{B}(v) = \frac{\sum_{u \in C} d_{uv}}{k}$$

$$\hat{B}(v) = B(c(v))$$

Hybrid Estimator !!

- Same computation/information as sampling/pivoting (k Dijkstras from a uniform sample)
- Use sample to estimate distances from v to "close" nodes
- Use pivot to estimate distances to "far" nodes

How to partition close/far ?

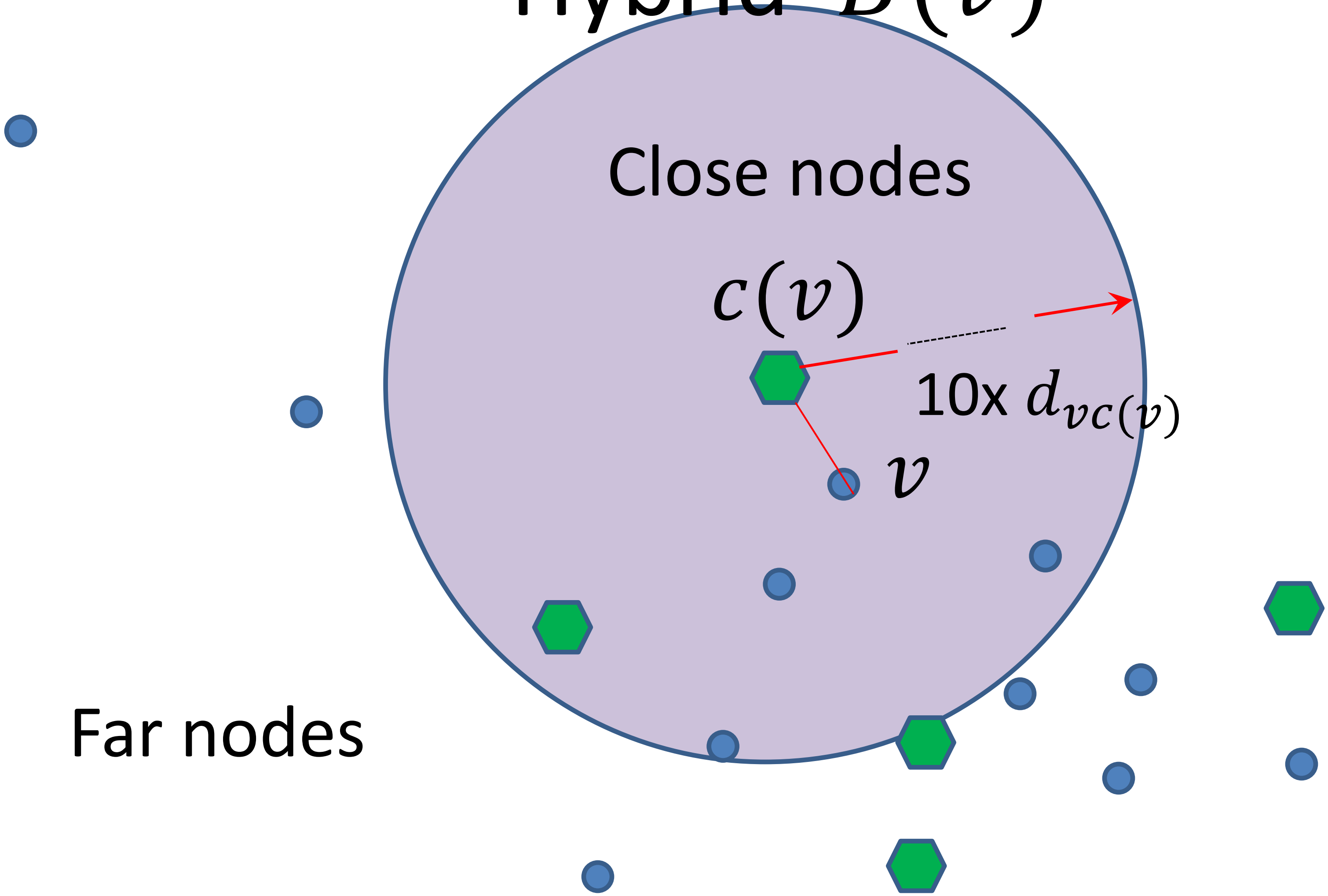
Idea: Look at distances of nodes from *the pivot* $c(v)$ (we have all these distances!)

Hybrid

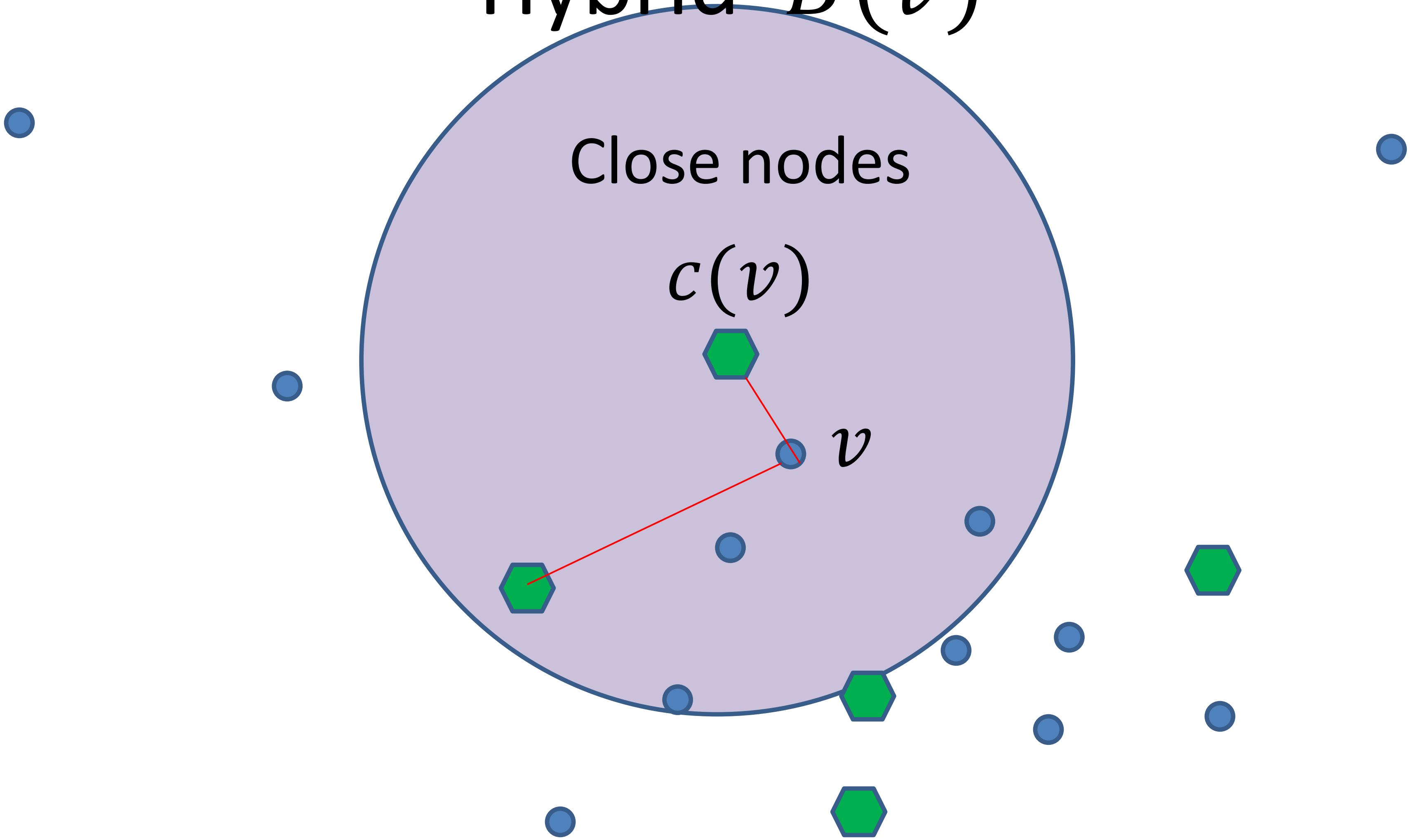
Partition nodes according to their distance to the pivot $c(v)$:

- Far nodes: Nodes $> d_{vc(v)}/\epsilon$ from pivot, use distance to pivot.
  - We have error at most $\pm d_{vc(v)}$ which is at most $1/(\frac{1}{\epsilon} - 1) \approx \epsilon$ contribution to relative error
- Close nodes: Nodes within $d_{vc(v)}/\epsilon$ from pivot, estimate using exact distances to *sampled* nodes
  - Intuition: We "cut off" the heavy tail that was bad for sampling

Hybrid $\hat{B}(v)$

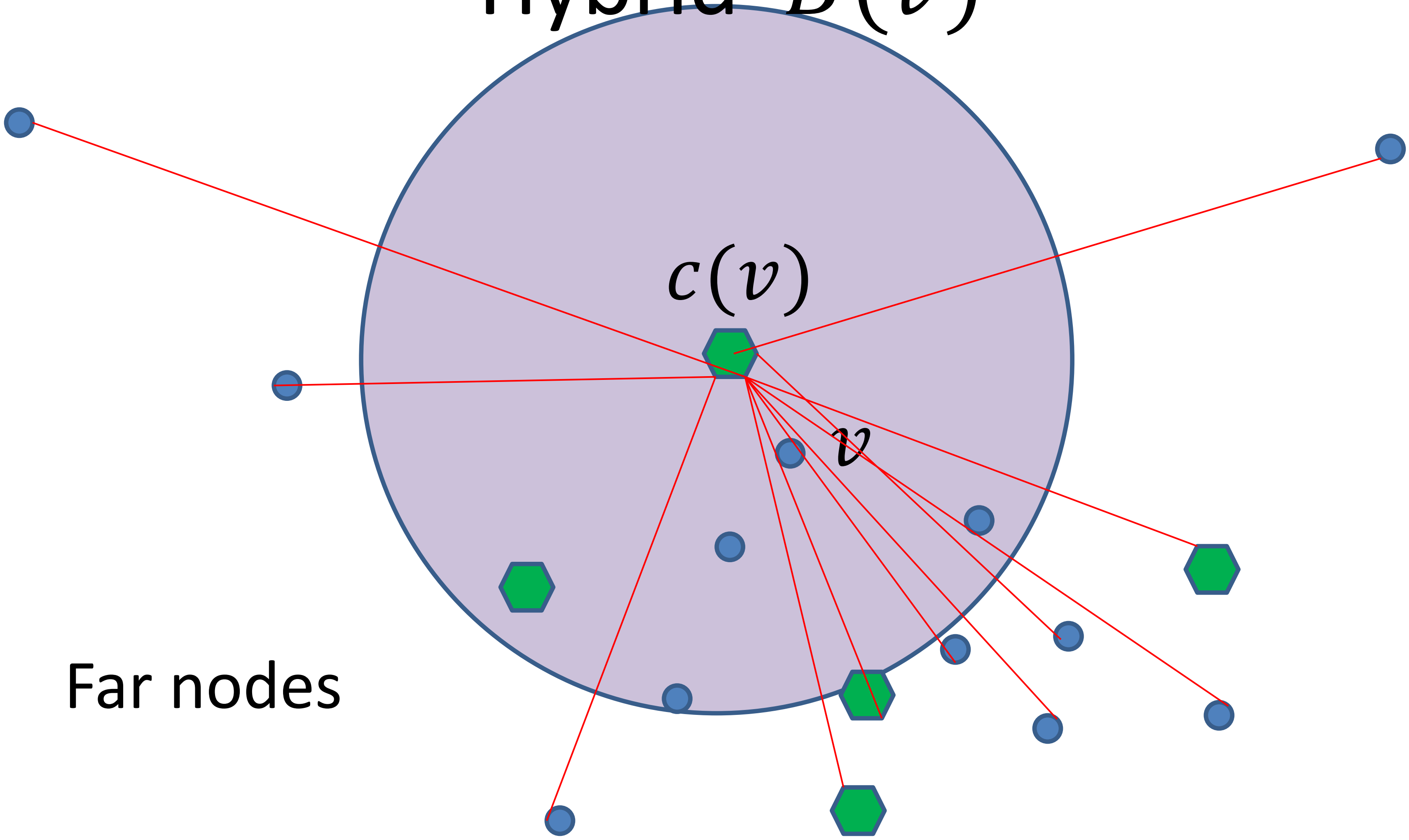


Hybrid $\hat{B}(v)$



6 close nodes (we know how many). Estimate using exact distances from v to the 2 close sampled nodes

Hybrid $\hat{B}(v)$



11 far nodes (we know which and how many). Estimate using distance from pivot c(v)

Analysis

How to set sample size k ?

Theory: (worse-case distance distribution)

$k \approx \epsilon^{-3}$ for NRMSE ϵ

($\times \log n$) for small error WHP for all nodes

Analysis (worst case)

- ❑ Far nodes: Nodes $> d_{vc(v)}/\epsilon$ from pivot$\approx \epsilon$ contribution to relative error
- ❑ Close nodes: We need $k \approx \epsilon^{-3}/2$ samples so that NRMSE (normalized standard error) at most ϵ

Idea: We estimate $\sum_{\{u\ close\}} d_{uv}$ by $\frac{\mathrm{n}}{\mathrm{k}}\sum_{\{u\ close\ in\ C\}} d_{uv}$

- Each $u \in C$ is sampled with $p = k/n \Rightarrow$ $var(\widehat{\sum_{\{u\ close\}} d_{\{uv\}}}) \leq \frac{n}{k}\sum_{\{u\ close\}} d_{\{uv\}}^2$
- Look at worst-case values $d_{uv} \in [0, \frac{d_{vc(v)}}{\epsilon}]$ that maximize $\sqrt{var}/\sum_u d_{uv}$

Analysis

How to set sample size k ?

<u>Theory</u>: (worse-case distance distribution)

$k \approx \epsilon^{-3}$

($\times \log n$) for small error WHP for all nodes

<u>Practice</u>: $k \approx \epsilon^{-2}$ works well.

What about the guarantees (want confidence intervals) ?

Adaptive Error Estimation

Idea: We use the information we have on the *actual* distance distribution to obtain tighter confidence bounds for our estimate than the worst-case bounds.

- Far nodes: Instead of using error $\pm d_{vc(v)}$, use sampled far nodes to determine if errors "cancel out." (some nodes closer to pivot $c(v)$ but some closer to v.
- Close nodes: Estimate population variance from samples.

Extension: Adaptive Error Minimization

For a given sample size (computation investment), and a given node, we can consider many thresholds for partitioning into closer/far nodes.

- We can compute an adaptive error estimate for each threshold (based on what we know on distribution).
- Use the estimate with smallest estimated error.

Efficiency

Given the kn distances from sampled nodes to all others, how do we compute the estimates efficiently?

- Partition "threshold" is different for different nodes with the same pivot (since it depends on distance to pivot).
- Can compute "suffix sums" of distances with Dijkstra from each pivot, to compute estimates for all nodes in $O(k)$ time per node

Scalability: Using +O(1)/node memory

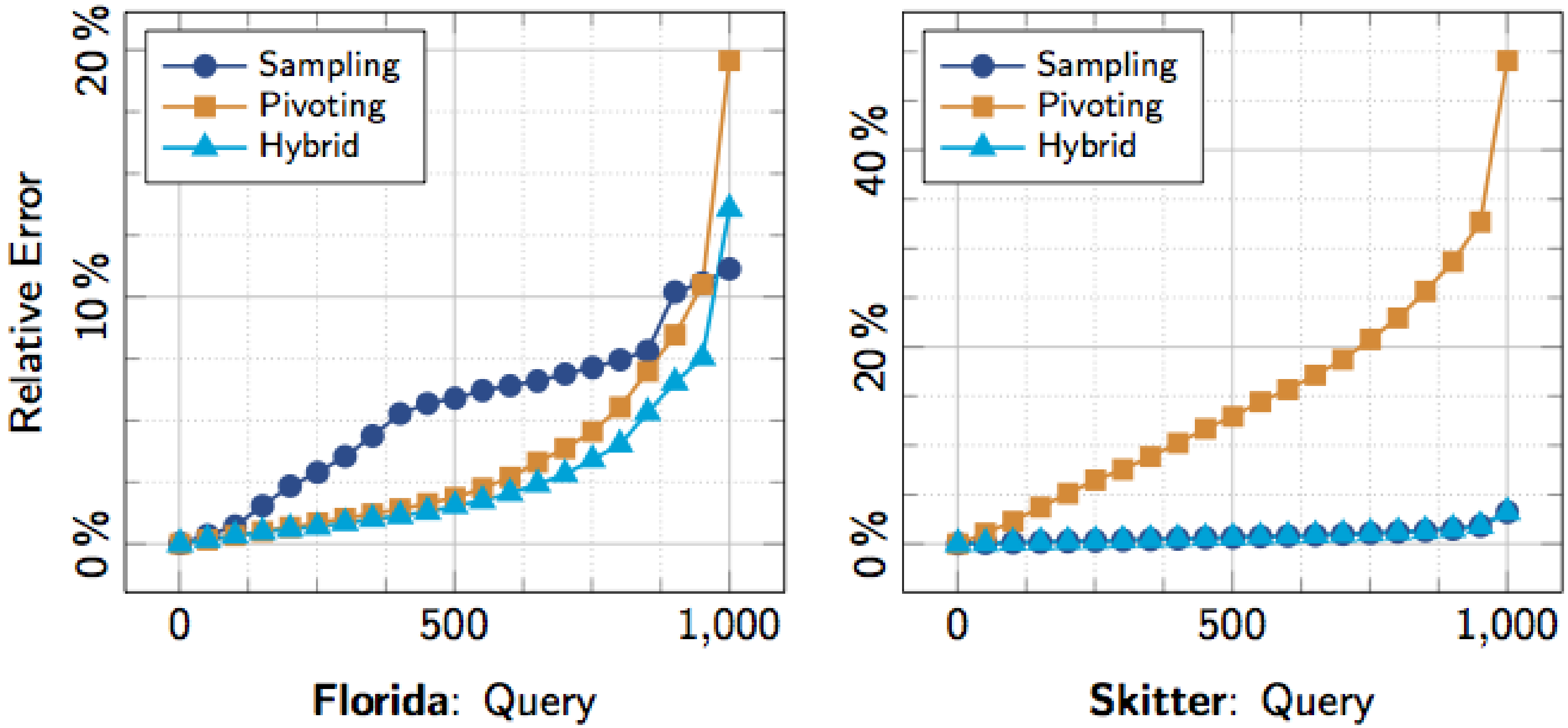
- We perform k Dijkstra's but do not want to store all kn distances.
- In our implementation, we reduce the additional storage to O(1) per node by first mapping nodes to their closest pivots. This is equivalent to performing one more Dijkstra.

Experimental Evaluation

type	instance	$\|E\|$ $[\cdot 10^3]$	Exact time $\approx$ [hrs]	Sampling err. [%]	Sampling time [sec]	Pivoting err. [%]	Pivoting time [sec]	Hybrid err. [%]	Hybrid time [sec]
road	fla-t	1 344	60	5.4	24.4	3.2	21.6	2.5	28.3
	usa-t	28 854	44 222	2.9	849.4	3.7	736.4	2.0	2 344.3
grid	grid20	2 095	71	4.3	26.5	3.5	26.8	2.9	29.2
triang	buddha-w	1 631	21	3.5	16.4	2.6	15.5	2.2	18.5
	del20-w	3 146	72	2.7	27.4	3.6	26.7	2.6	32.6
game	FrozenSea	2 882	38	3.0	22.1	4.1	20.2	2.1	24.0
sensor	rgg20-w	6 894	160	1.6	61.2	3.8	57.1	2.1	73.3
comp	Skitter	11 094	248	0.7	59.7	14.3	55.2	0.7	61.6
	MetroSec	21 643	270	0.6	52.1	2.3	47.5	0.6	53.2
social	rws20	3 146	114	0.9	45.6	3.0	41.3	0.9	49.4
	rba20	6 291	133	0.8	56.8	9.7	48.4	0.8	60.2
	Hollywood	56 307	227	1.0	86.5	14.6	81.8	1.0	85.7
	Orkut	117 185	2 973	1.7	377.4	7.2	367.6	1.7	376.4

Hybrid slightly slower, but more accurate than sampling or pivoting

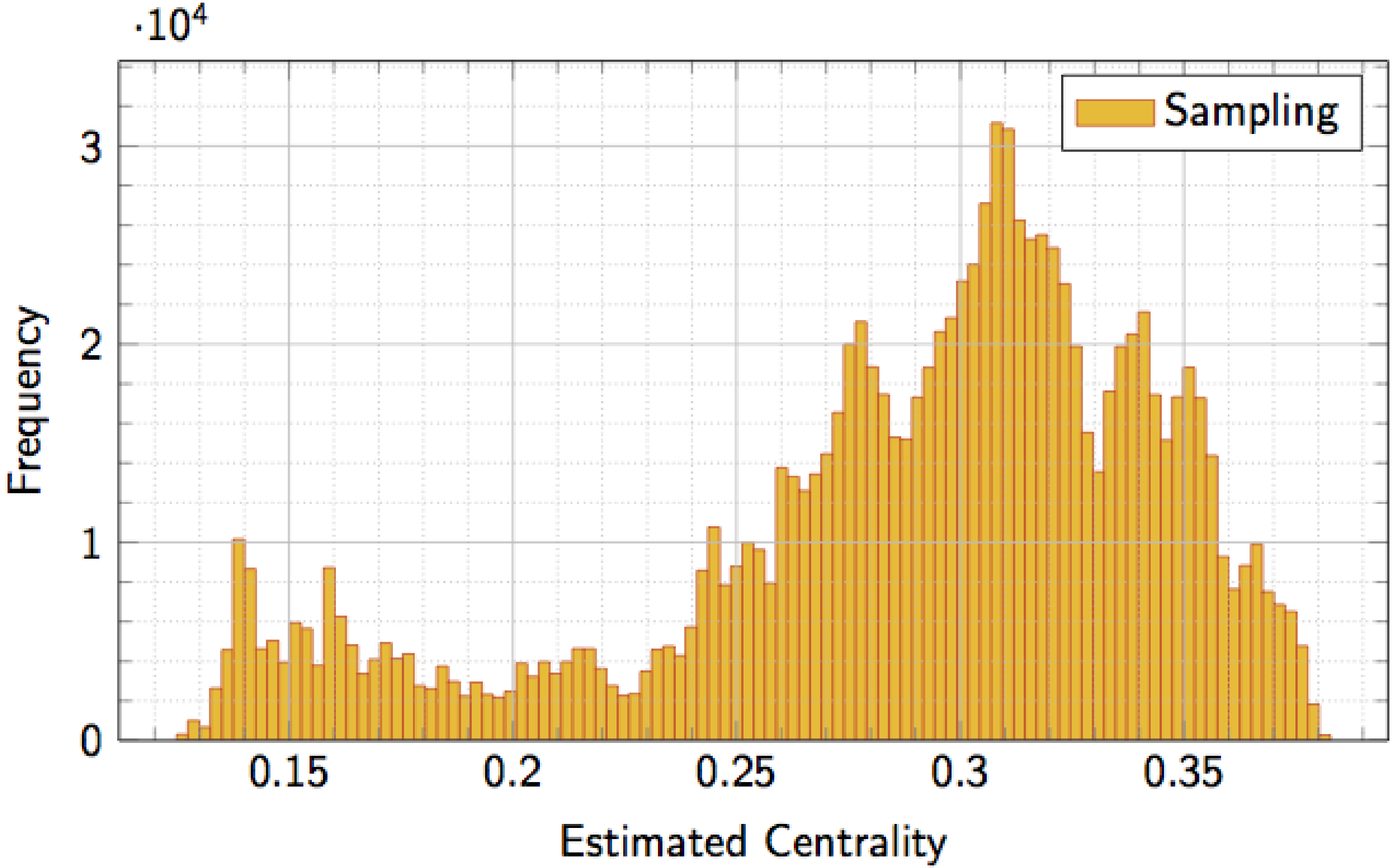
Experimental Evaluation



- Sampling: less accurate for "high diameter" graphs.
- Pivoting: less accurate for "low diameter" graph.
- Hybrid: Consistently good results (best of both).

Example Centrality Distribution

Graph: Road network of Florida with travel time metric.



Example Centrality Distribution

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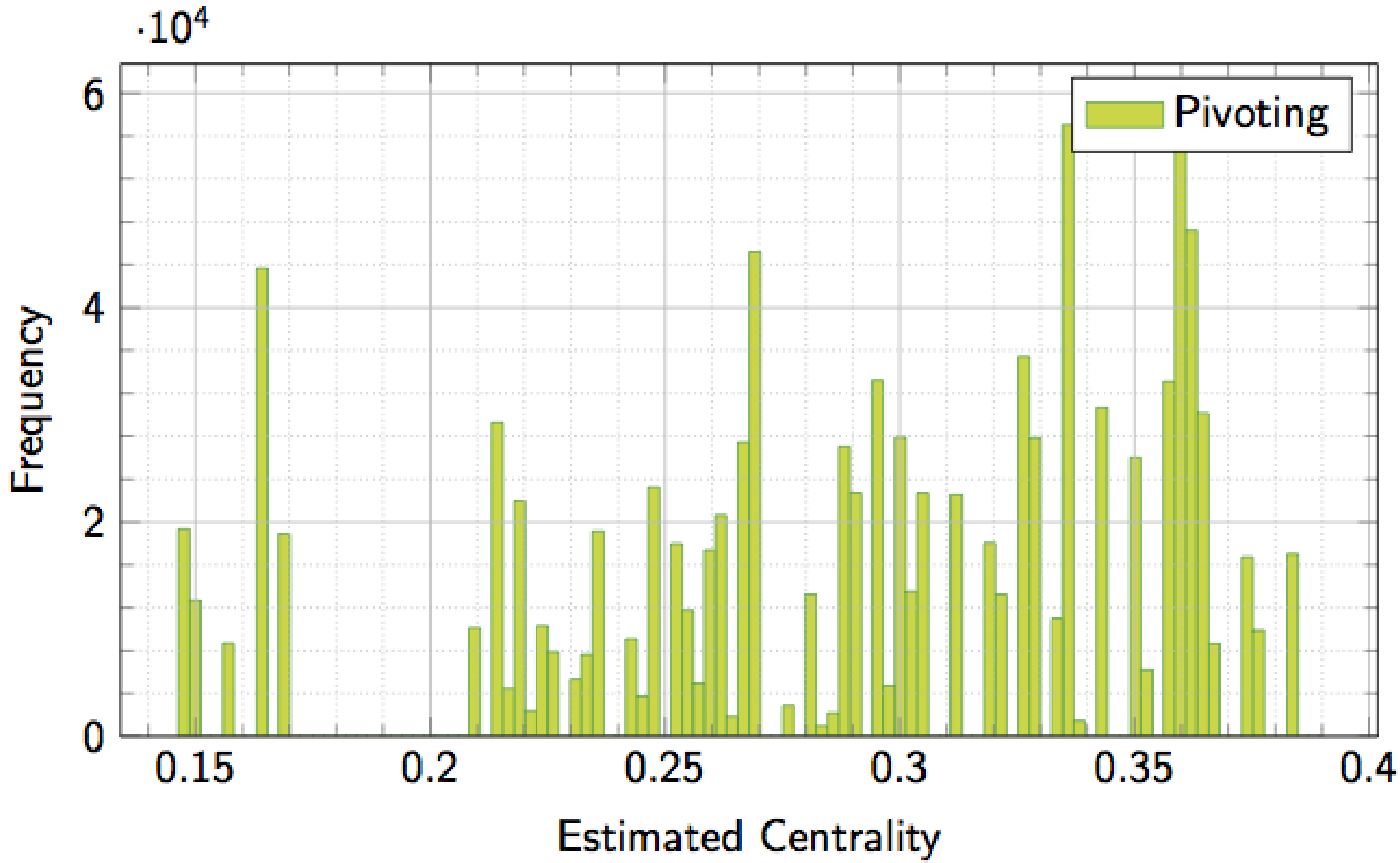

Example Centrality Distribution

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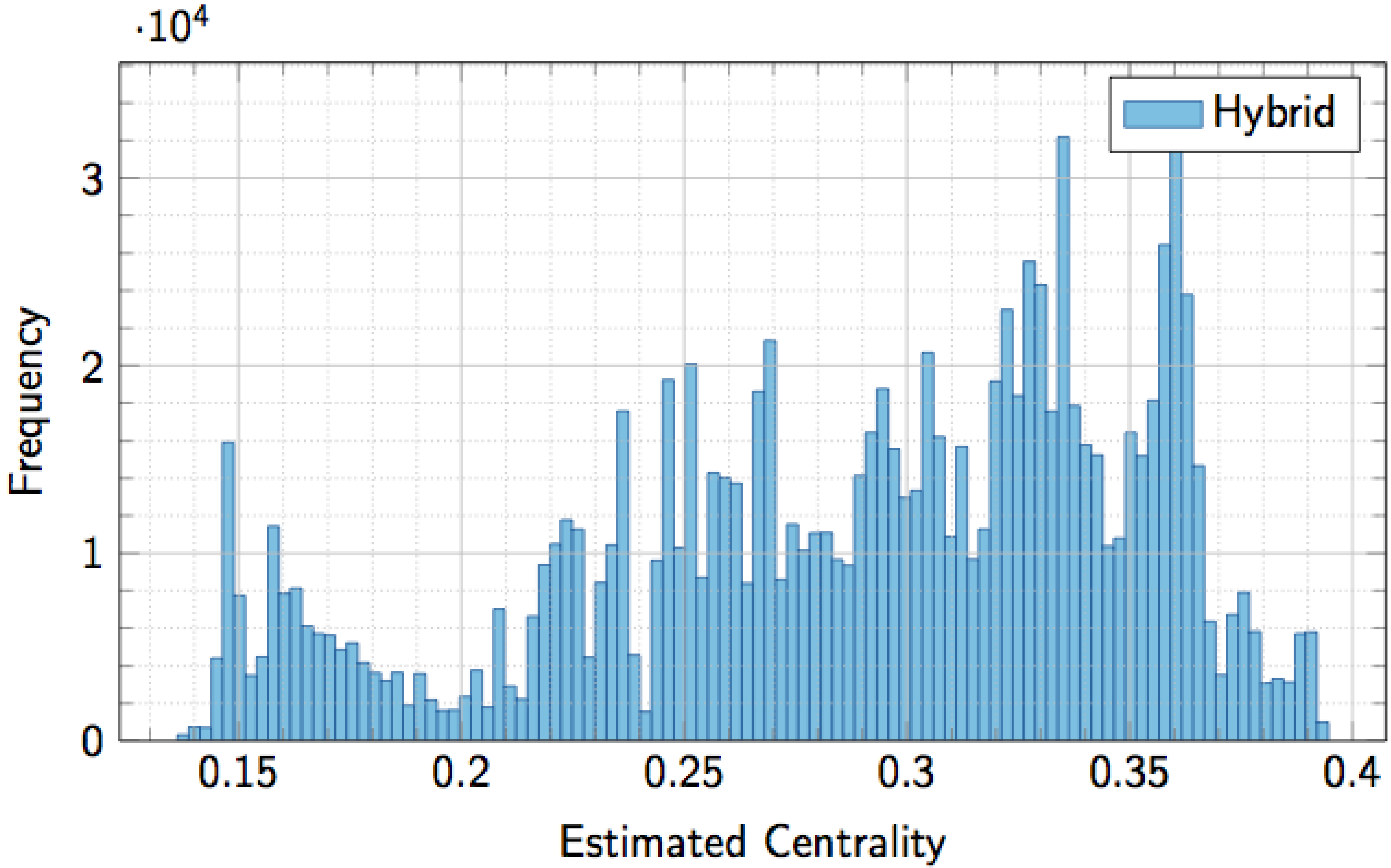

Directed graphs

(Classic Closeness) Centrality is defined as (inverse of) average distance to *reachable* (outbound distances) or *reaching* (inbound distances) nodes only.

- Sampling works (same properties) *when graph is strongly connected*.
- Pivoting breaks, even with strong connectivity. Hybrid therefore also breaks.
- When graph is not strongly connected, basic sampling also breaks – we may not have enough samples from each reachability set

We design a new sampling algorithm…

…Directed graphs

(Classic Closeness) Centrality is defined as (inverse of) average distance to *reachable* (outbound distances) or *reaching* (inbound distances) nodes only.

Algorithm computes for each node v its average distance to a uniform sample of k nodes from its reachability set. $\tilde{O}(k|G|)$ based on reachability sketches [C' 1994].

- Process nodes u in random permutation order
- Run Dijkstra from u, prune at nodes already visited k times

$\hat{B}(v) =$ sum of distances from visiting nodes / #visitors

Experimental Evaluation

type	instance	$\|V\|$ $[\cdot 10^3]$	$\|E\|$ $[\cdot 10^3]$	Exact time $\approx$ [h:m]	Sampling err. [%]	Sampling time [sec]
road	eur-t	18 010	42 189	28 399:47	3.2	655.9
web	NotreDame	326	1 470	0:54	2.4	1.5
	Indo	1 383	16 540	58:46	4.1	21.1
	Indochina	7 415	191 607	2 884:19	4.7	174.7
comp	Gnutella	63	148	0:02	2.8	0.6
social	Epinions	76	509	0:07	5.4	1.1
	Slashdot	82	870	0:18	2.2	2.2
	Flickr	1 861	22 614	227:01	4.3	65.1
	WikiTalk	2 394	5 021	22:01	0.5	5.4
	Twitter	457	14 856	28:16	1.2	26.1
	LiveJournal	4 848	68 475	2 757:01	1.9	276.8

Directed graphs: Reachability sketch based sampling is orders of magnitude faster with only a small error.

Extension: Metric Spaces

Basic hybrid estimator applies in *any metric space*: Using k single-source computations from a random sample, we can estimate centrality of all points with a small relative error.

Application: Centrality with respect to *Round-trip distances in directed strongly connected graphs*:

- Perform both a forward and back Dijkstra from each sampled node.
- Compute roundtrip distances, sort them, and apply estimator to that.

Extension: Node weights

Weighted centrality: Nodes are heterogeneous. Some are more important. Or more related to a topic. Weighted centrality emphasizes more important nodes.

$$B(v) = \frac{\sum_{u \in V} w(u) d_{uv}}{\sum_{u \in V} w(u)}$$

Variant of Hybrid with same strong guarantees uses a weighted (VAROPT) instead of a uniform nodes sample.

Closeness Centrality

- Classic (penalize for far nodes)

$$\boldsymbol{C(i) = (n-1) / \sum_j d_{ij}\, \beta(j)}$$

- Distance-decay (reward for close nodes)

$$\boldsymbol{C(i) = \sum_j \alpha(d_{ij})\, \beta(j)}$$

Different techniques required: All-Distances Sketches [C' 94] work for approximating distance-decay but not classic.

Summary

- Undirected graphs (and metric spaces): We combine sampling and pivoting to estimate classic closeness centrality of all nodes within a small relative error using k single-source computations.
- Directed graphs: Sampling based on reachability sketches
- Implementation: minutes on real-world graphs with hundreds of millions of edges

Future

- Estimate classic closeness centrality of all nodes within a small relative error using fewer single-source computations. Can do $k = \epsilon^{-2} \log n$ with adaptive choice of sources. Can we eliminate the union bound ?
- Can we do better in metric spaces (not confined to single source) ? Small dimension?
- Adaptive confidence bounds are applicable in many other problems. Should be used broadly.

Thank you!